IMPACT: International Journal of Research in Applied, Natural and Social Sciences (IMPACT: IJRANSS)
ISSN(P): 2347-4580; ISSN(E): 2321-8851
Vol. 5, Issue 3, Mar 2017, 151-156

LAPLACE TRANSFORM OF POWER SERIES

SHIFERAW GEREMEW KEBEDE

Department of Mathematics, Madda Walabu University, Bale Robe, Ethiopia

ABSTRACT

The authors establish a set of presumably new results, which provide Laplace transform of power series. So in this paper the author try to evaluate Laplace transform of some challenging functions by expressing them as a sum of infinitely terms. Hence, the method is useful to and the Laplace transform of functions that do not have elementary Laplace transformations.

KEYWORDS: Laplace Transforms, Power Series

INTRODUCTION

The Laplace transform is one of the most important integral transforms. Because of a number of special properties, it is very useful in studying linear differential equations.

The Laplace transform is named after mathematician and astronomer Pierre-

Simon Laplace, who used a similar transform (now called the z-transform) in his work on probability theory.[2] The current widespread use of the transform (mainly in engineering) came about during and soon after World War II [3] although it had been used in the 19th century by Abel, Lerch, Heaviside, and Bromwich.

The early history of methods having some similarity to Laplace transform is as follows. From 1744, Leonhard Euler investigated integrals of the form as solutions of differential equations but did not pursue the matter very far. [4] Joseph Louis Lagrange was an admirer of Euler and, in his work on integrating probability density functions, investigated expressions of the form which some modern historians have interpreted within modern Laplace transform theory.[5][6][clarification needed]

These types of integrals seem first to have attracted Laplace's attention in

1782 where he was following in the spirit of Euler in using the integrals themselves as solutions of equations.[7] However, in 1785, Laplace took the critical step forward when, rather than just looking for a solution in the form of an integral, he started to apply the transforms in the sense that was later to become popular. He used an integral of the form akin to a Mellin transform, to transform the whole of a difference equation, in order to look for solutions of the transformed equation. He then went on to apply the Laplace transform in the same way and started to derive some of its properties, beginning to appreciate its potential power. [8]

Laplace also recognized that Joseph Fourier's method of Fourier series for solving the diffusion equation could only apply to a limited region of space because those solutions were periodic. In 1809, Laplace applied his transform to find solutions that diffused indefinitely in space.[9]

Definition

The Laplace transform of the function $f: (0, \infty) \to \mathbb{R}$ is given by:

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

Where $s \in \mathbb{R}$ is any real number such that the integral above converges?

Definition

A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

Definition

Maclaurin Series expansion of the function $f(x)$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

Definition

The gamma function, whose symbol $\Gamma(s)$ is defined; when $s > 0$ by the formula:

$$\Gamma = \int_0^\infty e^{-x} x^{n-1} dx$$

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Theorem: (Laplace Transform of Power Series)

If $f(t)$ has a power series expansion of at c, where is c any constant$c \in \mathbb{R}$;

Its Taylor's series expansion is:

$$f(t) = \sum_{n=0}^{\infty} a_n (t-c)^n$$

Then; the Laplace transform of $f(t)$ is given in the form of power series as:

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}\left(\sum_{n=0}^{\infty} a_n (t-c)^n\right)$$

$$= \sum_{n=0}^{\infty} a_n \frac{1}{e^{sc}} \frac{\Gamma(n+1)}{s^{n+1}}$$

If $f(t)$ has a power series expansion of at0, then the power series expansion of $f(t)$ is given by:

$$f(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then the Laplace transform of $f(t)$ is defined by:

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}\left(\sum_{n=0}^{\infty} a_n t^n\right)$$

$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{s^{n+1}}$$

Proof

Suppose $f(t)$ has a power series expansion at$c$.

$f(t) = \sum_{n=0}^{\infty} a_n t^n$

Then:

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}(\textstyle\sum_{n=0}^{\infty} a_n (t-c)^n)$$

$$= \int_0^{\infty} (e^{-st} \textstyle\sum_{n=0}^{\infty} a_n (t-c)^n)\, dt$$

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$$= \textstyle\sum_{n=0}^{\infty} a_n \int_0^{\infty} e^{-st} (t-c)^n\, dt$$

$let, x = t - c, \Rightarrow t = x + c \; and \; dt = dx$

$$\Rightarrow \mathcal{L}\big(f(t)\big) = \mathcal{L}(\textstyle\sum_{n=0}^{\infty} a_n (t-c)^n)$$

$$= \textstyle\sum_{n=0}^{\infty} a_n \int_0^{\infty} e^{-s(x+c)} x^n\, dx$$

$$= \textstyle\sum_{n=0}^{\infty} a_n \int_0^{\infty} e^{-sx} e^{-sc} x^n\, dx$$

$$= \textstyle\sum_{n=0}^{\infty} a_n e^{-sc} \int_0^{\infty} e^{-sx} x^n\, dx$$

$$= \textstyle\sum_{n=0}^{\infty} a_n e^{-sc} \frac{\Gamma(n+1)}{s^{n+1}}$$

$$= \textstyle\sum_{n=0}^{\infty} a_n \frac{1}{e^{sc}} \frac{\Gamma(n+1)}{s^{n+1}}$$

Note: In particular, for $n = 1,2,3, \ldots$

$\Gamma(n+1) = n!$

In this case:

$$\mathcal{L}(\textstyle\sum_{n=0}^{\infty} a_n (t-c)^n) = \textstyle\sum_{n=0}^{\infty} a_n \frac{1}{e^{sc}} \frac{n!}{s^{n+1}}$$

Suppose $f(t)$ has a power series expansion at 0, is given by:

$f(t) = \sum_{n=0}^{\infty} a_n t^n$

By using the definition of Laplace transforms,

$$\mathcal{L}\big(f(t)\big) = \mathcal{L}(\textstyle\sum_{n=0}^{\infty} a_n t^n)$$

$$= \int_0^{\infty} (e^{-st} \textstyle\sum_{n=0}^{\infty} a_n t^n)\, dt$$

$$= \int_0^{\infty} (\textstyle\sum_{n=0}^{\infty} e^{-st} a_n t^n)\, dt$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} (e^{-st} a_n t^n)\, dt$$

$$= \sum_{n=0}^{\infty} a_n \int_0^{\infty} e^{-st} t^n\, dt$$

$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{s^{n+1}}$$

Note: In particular, for $n = 1,2,3, \dots$

$\Gamma(n+1) = n!$

Hence,

$\mathcal{L}(\sum_{n=0}^{\infty} a_n t^n) = \sum_{n=0}^{\infty} a_n \frac{\mathrm{n!}}{s^{n+1}}$

Example1

Find the Laplace transform of $f(t) = e^{t^2}$, after expanding to power series form

Solution

The power series expansion of

$f(t) = e^{t^2} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!}$

$$= \mathcal{L}\left(\sum_{n=0}^{\infty} \frac{t^{2n}}{n!}\right)$$

$$= \int_0^{\infty} \left(e^{-st} \sum_{n=0}^{\infty} \frac{t^{2n}}{n!}\right) dt$$

$$= \int_0^{\infty} \left(\sum_{n=0}^{\infty} e^{-st} \frac{t^{2n}}{n!}\right) dt$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} \left(e^{-st} \frac{t^{2n}}{n!}\right) dt$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} (e^{-st} t^{2n})\, dt$$

Let $2n = m$,

$$\Rightarrow \mathcal{L}\left(\sum_{n=0}^{\infty} \frac{t^{2n}}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} (e^{-st} t^m)\, dt$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(m+1)}{s^{m+1}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(2n+1)}{s^{2n+1}}$$

Example 2

Find the Laplace transform of$f(t) = \frac{\sin t}{t}$, after expanding to power series form

Solution

The power series expansion of:

$$f(t) = \frac{\sin t}{t}$$

$$= \frac{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots}{t}$$

$$= 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$$

Hence, $\mathcal{L}\big(f(t)\big) = \mathcal{L}\left(\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}\right)$

$$= \int_0^{\infty} e^{-st} \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}\right) dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} e^{-st} \frac{t^{2n}}{(2n+1)!} dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2n+1)}{(2n+1)s^{2n+1}}$$

CONCLUSIONS

The results on Laplace transform of power series are summarized as follows; Some functions like e^{t^2}, $\frac{\sin t}{t}$ and son on are difficult to get their Laplace transform. Hence it is possible to transform such functions to Laplace transform by expanding them into power series form as:

$$\mathcal{L}\big(f(t)\big) = \sum_{n=0}^{\infty} a_n \frac{1}{e^{sc}} \frac{\Gamma(n+1)}{s^{n+1}}$$

Where, $f(t) = \sum_{n=0}^{\infty} a_n (t-c)^n$, $c \in \mathbb{R}$.

$$\mathcal{L}\big(f(t)\big) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{s^{n+1}}$$

Where, $f(t) = \sum_{n=0}^{\infty} a_n t^n$

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